Technical Appendix for: Real Financial Contracts - no depreciation in financial contracts

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Contents

1 The Households 2

2 The Financial Contract 3

3 The intermediate goods producing firms 3
   3.1 The optimization problem 3
   3.2 Recursive formulation of FOC for $k_t$ 6
   3.3 The aggregated profit for intermediate goods producing firms 7

4 The Financial Intermediaries 7
   4.1 Balance sheet of the representative bank and its net-worth 8
   4.2 The value function for the representative bank 10
   4.3 The incentive constraint for the bank 12

5 Insurance company 13

6 Capital Producing Firms 13
   6.1 The first order conditions: 15
   6.2 Simplified formulation without adjustment costs 16

7 Market Clearing 17

8 Summarizing 18

9 The deterministic steady state 19

10 Expression for the external finance premium and leverage 25

11 Steady state analysis: capital level and firms profit 27

12 Calibration of the proportional bank tax 28

13 Steady state comparison with the standard RBC model 29
1 The Households

Each household is composed of workers and bankers. It is assumed that, in any period, a fraction \( 1 - f \) of household members are workers and the remaining fraction \( f \) are bankers. A banker in a given period continues to be a banker next period with probability \( b \), which is independent of history. The dynamic optimization problem faced by the representative household is of the form:

\[
\max_{c_t, x_{t+1}, h_t, k_{t+1}, \forall t \geq 0} \quad U_t = E_t \sum_{l=0}^{\infty} \beta^l \left( \frac{(c_{t+l} - bc_{t+l-1})^{1-\sigma_c}}{1-\sigma_c} - \phi_2 h_{t+l}^{1+\phi_1} \right)
\]

s.t.:

- a budget constraint
- a no-Ponzi-game condition

\( c_t, h_t \geq 0 \) and \( R_t \geq 1 \ \forall t \geq 0 \)

where \( u_t \) represents a shock to household preferences. The lagrangian for this problem reads:

\[
\mathcal{L} = E_t \sum_{l=0}^{\infty} \beta^l \left( \frac{(c_{t+l} - bc_{t+l-1})^{1-\sigma_c}}{1-\sigma_c} - \phi_2 h_{t+l}^{1+\phi_1} \right) \]

\[
+ E_t \sum_{l=0}^{\infty} \beta^l \lambda_{t+l} [x_{t+l} + w_{t+l} h_{t+l} + \text{profit}_{t+l} - D_{t+t+1} x_{t+l+1} - c_{t+l}]
\]

**FOC (Kuhn-Tucker conditions)**

We look for a solution in the interior, i.e. \( c_t, x_{t+1}, h_t > 0 \ \forall t \geq 0 \)

1. **Consumption, \( c_t \):**

\[
\frac{\partial \mathcal{L}}{\partial c_t} = E_t \left[ (c_t - bc_{t-1})^{-\sigma_c} - \beta b (c_{t+1} - bc_t)^{-\sigma_c} - \lambda_t \right] = 0
\]

\[\downarrow\]

\[
\lambda_t = E_t \left[ (c_t - bc_{t-1})^{-\sigma_c} - \beta b (c_{t+1} - bc_t)^{-\sigma_c} \right]
\] (1)

2. **State-contingent claims in period \( t+1, x_{t+1} (s) \):**

\[
\frac{\partial \mathcal{L}}{\partial x_{t+1}(s)} = \text{prob}(s) [\beta \lambda_{t+1} (s) - D_{t+t+1} (s) \lambda_t] = 0
\]

\[\downarrow\]

\[
D_{t+t+1} (s) = \beta \frac{\lambda_{t+1} (s)}{\lambda_t}
\]

Surpressing the state-dimension we may write

\[
D_{t+t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \ \forall \text{states}
\]

Pricing a 1 period real zero coupon bond gives

\[
1 = E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} R_t \right]
\] (2)

3. **The labor supply, \( h_t \):**

\[
\frac{\partial \mathcal{L}}{\partial h_t} = -\phi_2 h_t^{\phi_1} + \lambda_t w_t = 0
\]

\[\downarrow\]

\[
\phi_2 h_t^{\phi_1} = \lambda_t w_t
\] (3)
2 The Financial Contract

3 The intermediate goods producing firms

3.1 The optimization problem

This section derives the necessary and sufficient first-order-conditions for the \(i\)'th firm’s optimization problem. In each period, firm \(i\) needs to borrow from the financial intermediary in order to acquire capital for production. Since \(p^k_t\) is the price of capital, the firm must therefore borrow \(k_{i,t}p^k_t\).

The financial contract that allows the firm to borrow from the intermediary is defined as follows. A contract written at date \(t\) specifies an amount of borrowing chosen by the firm, \(k_{i,t}p^k_t\), and a per-period gross rate of return, \(R^k_t \equiv (1 + r^k_t)\) where \(r^k_t \geq 0\). It is further assumed that the firm clears its gains and losses on capital each period.

We assume a Calvo structure for pinning down the average duration of the financial contracts. Accordingly the \(i\)'th firm is only allowed to optimize its level of capital each period with probability \(1 - \alpha_k\). Hence, the average duration of the financial contract is \(\frac{1}{1 - \alpha_k}\).

We assume the following timing for the firm’s decisions:

- At the beginning of period \(t\), the firm finds out if it is allowed to reoptimize the level of capital or not.
- If it cannot reoptimize, \(k_{i,t} = k_{i,t-1}\) and the firm thus borrows \(p^k_{t-1}k_{i,t-1}\).
- If the firm reoptimizes at \(t\), it chooses the optimal level of \(k_{i,t}\), which we denote by \(\tilde{k}_{i,t}\). In this case the firm borrows \(p^k_t\).
- The firm then uses the credit line obtained from the financial intermediary to buy capital in a competitive market. It additionally rents labor (we assume that the wage bill is paid at the end of the period) and produces the homogeneous intermediate good;
- At the end of period \(t\) the firm clears the losses associated to the capital and repays the loans obtained from the financial intermediaries.

The expression for profit for the \(i\)'th firm is thus (this expression reduces below for the optimizing firms)

\[
\pi^{int}_{i,t} = a_t k^\theta_{i,t} h_{i,t}^{1-\theta} + (p^k_t k_{i,t} - p^k_t k_{i,t}) \\
+ (p^k_t k_{i,t}(1 - \omega) - (1 + r^k_t) p^k_t k_{i,t} - w_t h_{i,t}) \\
 \text{production income} \quad \text{borrow} \quad \text{buy capital} \\
\text{sell capital net of service fee} \quad \text{repay loan} \quad \text{pay labour}
\]

\[
\tilde{\pi}^{int}_{i,t} = a_t + j k^\theta_{i,t} h_{i,t}^{1-\theta} - (r^k_{i,t} + \omega) p^k_t k_{i,t} - w_t h_{i,t}
\]

So the problem for the \(i\)'th firm is

\[
\text{Max}_{\beta_t, k_{i,t}, h_{i,t}, \forall t \geq 0} \quad \text{profit}_{i,t} = E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_t}{\lambda_t} \left( a_t + j k^\theta_{i,t+j} h_{i,t+j}^{1-\theta} - (r^k_{i,t+j} + \omega) p^k_t k_{i,t+j} - w_t h_{i,t+j} \right)
\]

St. sticky loans decisions

- a no-Ponzi-game condition
- \(h_{i,t}, k_{i,t} \geq 0 \quad \forall t \geq 0\)

**FOC** (Kuhn-Tucker conditions):

**Labor, \(h_{i,t}\):**

\[
\frac{\partial \pi^{int}_{i,t}}{\partial h_{i,t}} = a_t (1 - \theta) k^\theta_{i,t} h_{i,t}^{1-\theta} - w_t = 0
\]

\[
a_t (1 - \theta) \left( k^\frac{\beta_{i,t}}{h_{i,t}} \right)^{\theta}_t = w_t
\]

**Profit, \(k_{i,t}\):**
\[ \begin{align*}
&\left( \frac{k_{i,t}}{h_{i,t}} \right)^{\theta} = \frac{w_t}{a_t(1-\theta)} \\
&\left( \frac{k_{i,t}}{n_{i,t}} \right)^{\frac{1}{\theta}} = \left( \frac{w_t}{a_t(1-\theta)} \right)^{\frac{1}{\theta}}
\end{align*} \]

Therefore, due to a perfectly competitive labour market and only aggregated productivity shocks, the ratio \( \frac{k_{i,t}}{n_{i,t}} \) will be identical for all firms.

Firms that reoptimize at \( t \) will choose \( \tilde{k}_{i,t} \). These firms will also choose labor so that the capital-labor ratio equalizes throughout all firms. Letting \( \tilde{h}_{i,t+j|t} \) denote the period \( t+j \) labor demand for a firm that last reoptimizes in period \( t \) we get:

\[ \tilde{h}_{i,t+j|t} = \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1}{\theta}} \tilde{k}_{i,t} \]

**Capital, \( k_{i,t} \):**

We first note that the expression for profits \( \pi_{i,t}^{int} \) simplify as follows for the optimizing firms:

\[ \pi_{t+j|i} = a_{t+j} \tilde{k}_{t}^{1-\theta} \tilde{h}_{t+j|t} - (r_{t}^{L} + \omega) p_{t}^{L} \tilde{k}_{t} - w_{t+j} \tilde{h}_{t+j|t} \]

\[ \pi_{t+j|i} = a_{t+j} \tilde{k}_{t}^{1-\theta} \tilde{h}_{t+j|t} - (r_{t}^{L} + \omega) p_{t}^{L} \tilde{k}_{t} - w_{t+j} \tilde{h}_{t+j|t} \]

We have omitted the index \( i \) for the firms as all the optimizing firms are identical. Thus

\[ \max_{k_{i,t}} \pi_{i,t}^{opt} = E_{t} \sum_{j=0}^{\infty} \alpha_{t+j}^{\beta} \frac{\lambda_{t+j}}{\lambda_{t}} \left( a_{t+j} \tilde{k}_{t}^{1-\theta} \tilde{h}_{t+j|t} - (r_{t}^{L} + \omega) p_{t}^{L} \tilde{k}_{t} - w_{t+j} \tilde{h}_{t+j|t} \right) \]

The first order condition for \( \pi_{t+j|i}^{int} \) with respect to \( \tilde{k}_{i,t} \) is

\[ \frac{\partial \pi_{t+j|i}^{int}}{\partial \tilde{k}_{i,t}} = E_{t} \sum_{j=0}^{\infty} \alpha_{t+j}^{\beta} \frac{\lambda_{t+j}}{\lambda_{t}} \left[ \theta a_{t+j} \left( \tilde{k}_{t} \right)^{\theta-1} \tilde{h}_{t+j|t}^{1-\theta} - (r_{t}^{L} + \omega) p_{t}^{L} \right] = 0 \]

We can simplify this expression and eliminate \( h_{t+j|t} \) using \( \tilde{h}_{t+j|t} = \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1}{\theta}} \tilde{k}_{t} \). Hence

\[ E_{t} \sum_{j=0}^{\infty} \alpha_{t+j}^{\beta} \frac{\lambda_{t+j}}{\lambda_{t}} \left[ \theta a_{t+j} \left( \tilde{k}_{t} \right)^{\theta-1} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} \left( \tilde{k}_{t} \right)^{1-\theta} - (r_{t}^{L} + \omega) p_{t}^{L} \right] = 0 \]

\[ E_{t} \sum_{j=0}^{\infty} \alpha_{t+j}^{\beta} \frac{\lambda_{t+j}}{\lambda_{t}} \left[ \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} - (r_{t}^{L} + \omega) p_{t}^{L} \right] = 0 \]

\[ E_{t} \sum_{j=0}^{\infty} \alpha_{t+j}^{\beta} \frac{\lambda_{t+j}}{\lambda_{t}} \left[ \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} - (r_{t}^{L} + \omega) p_{t}^{L} \right] = 0 \]

Notice that the marginal product of capital is given by

\[ \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} = \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} = \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} = \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} \]

\[ = \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} \theta w_{t+j} \]

Notice that the marginal product of capital is given by

\[ \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} = \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} = \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} = \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} \]

\[ = \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right)^{-\frac{1-\theta}{\theta}} \theta w_{t+j} \]
In what follows we will find a recursive form for this equation, but first we will derive law of motions for aggregate capital and labor.

**Aggregation of the capital stock:**
The aggregate capital stock is then given by

\[ k_t = \int_0^1 k_{i,t} d\bar{i} \]

\[ = (1 - \alpha_k) \bar{k}_t + (1 - \alpha_k) \alpha_k \bar{k}_{t-1} + ... \]

\[ = (1 - \alpha_k) \sum_{i=0}^{\infty} \alpha_k^i \bar{k}_{t-i} \]

Here, we use the fact that the number of firms is by construction large, so there is a fraction \(1 - \alpha_k\) of firms reoptimizing their capital holdings each period and the remaining fraction use the capital value from the previous period. Then we note that:

\[ k_t = (1 - \alpha_k) \bar{k}_t + \alpha_k k_{t-1} \]

(4)

**Aggregation of the labor demand:**
The aggregate level of labour is given by

\[ h_t = \int_0^1 h_{i,t} d\bar{i} \]

\[ = (1 - \alpha_k) \bar{h}_{t|t} + \int_{Z_{t-1}} h_{i,t} d\bar{i} \]

\[ = (1 - \alpha_k) \bar{h}_{t|t} + \left( \frac{w_t}{a_t(1-\theta)} \right)^{-\frac{1}{\theta}} \int_{Z_{t-1}} k_{i,t} d\bar{i} \]

where \(Z_{t-1}\) represents the set of firms that last reoptimized in period \(t-1\) or before and we use \(h_{i,t} = \left( \frac{w_t}{a_t(1-\theta)} \right)^{-\frac{1}{\theta}} k_{i,t}\). By the law of large numbers, aggregate labor evolves according to

\[ h_t = (1 - \alpha_k) \bar{h}_{t|t} + \left( \frac{w_t}{a_t(1-\theta)} \right)^{-\frac{1}{\theta}} \alpha_k k_{t-1} \]

We also know that, for the firms optimizing in the current period, the labor demand is given by

\[ \bar{h}_{t|t} = \left( \frac{w_t}{a_t(1-\theta)} \right)^{-\frac{1}{\theta}} \bar{k}_t \]

Combining the two equations above:

\[ h_t = (1 - \alpha_k) \left( \frac{w_t}{a_t(1-\theta)} \right)^{-\frac{1}{\theta}} \bar{k}_t + \left( \frac{w_t}{a_t(1-\theta)} \right)^{-\frac{1}{\theta}} \alpha_k k_{t-1} \]

\[ = \left( \frac{w_t}{a_t(1-\theta)} \right)^{-\frac{1}{\theta}} \left[ (1 - \alpha_k) \bar{k}_t + \alpha_k k_{t-1} \right] \]

(5)

Notice that this equation actually follows directly from the first order conditions for labor as this condition holds for ALL firms.
3.2 Recursive formulation of FOC for $\tilde{k}_t$

We need to find a recursion for

$$E_t \sum_{j=0}^{+\infty} (\alpha \beta)^j \frac{\lambda_{t+j}}{\lambda_t} \left[ \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right) ^{-\frac{1}{\theta}} - (r_t^L + \omega) p_t^k \right] = 0$$

where we have defined

$$z_{1,t} := E_t \sum_{j=0}^{+\infty} (\alpha \beta)^j \frac{\lambda_{t+j}}{\lambda_t} \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right) ^{-\frac{1}{\theta}}$$

$$z_{2,t} := E_t \sum_{j=0}^{+\infty} (\beta \alpha)^j \frac{\lambda_{t+j}}{\lambda_t}$$

**Recursion for $z_{1,t}$**

$$z_{1,t} = E_t \sum_{j=0}^{+\infty} (\alpha \beta)^j \frac{\lambda_{t+j}}{\lambda_t} \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right) ^{-\frac{1}{\theta}}$$

$$= \theta a_t \left( \frac{w_t}{a_t(1-\theta)} \right) ^{-\frac{1}{\theta}} + E_t \sum_{j=0}^{+\infty} (\alpha \beta)^j \frac{\lambda_{t+j}}{\lambda_t} \theta a_{t+j} \left( \frac{w_{t+j}}{a_{t+j}(1-\theta)} \right) ^{-\frac{1}{\theta}}$$

$$z_{1,t} = \theta a_t \left( \frac{w_t}{a_t(1-\theta)} \right) ^{-\frac{1}{\theta}} + E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} z_{1,t+1} \alpha_k \right]$$

(7)

because

$$z_{1,t+1} = E_{t+1} \sum_{j=0}^{+\infty} (\alpha \beta)^j \frac{\lambda_{t+1+j}}{\lambda_{t+1}} \theta a_{t+1+j} \left( \frac{w_{t+1+j}}{a_{t+1+j}(1-\theta)} \right) ^{-\frac{1}{\theta}}$$

$$= E_{t+1} \sum_{i=1}^{+\infty} (\alpha \beta)^{i-1} \frac{\lambda_{t+i}}{\lambda_{t+1}} \theta a_{t+i} \left( \frac{w_{t+i}}{a_{t+i}(1-\theta)} \right) ^{-\frac{1}{\theta}}$$

change of index: $i = 1 + j$

$$E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} z_{1,t+1} \alpha_k \right] = E_{t+1} \sum_{i=1}^{+\infty} (\alpha \beta)^{i} \frac{\lambda_{t+i}}{\lambda_{t+1}} \theta a_{t+i} \left( \frac{w_{t+i}}{a_{t+i}(1-\theta)} \right) ^{-\frac{1}{\theta}}$$

Recursion for $z_{2,t}$

$$z_{2,t} = E_t \sum_{j=0}^{+\infty} (\beta \alpha)^j \frac{\lambda_{t+j}}{\lambda_t}$$

$$= 1 + E_t \sum_{j=0}^{+\infty} (\beta \alpha)^j \frac{\lambda_{t+j}}{\lambda_t}$$

$$z_{2,t} = 1 + E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} (\beta \alpha_k) z_{2,t+1} \right]$$

(8)

because

$$z_{2,t+1} = E_{t+1} \sum_{j=0}^{+\infty} (\beta \alpha)^j \frac{\lambda_{t+1+j}}{\lambda_{t+1}}$$

6
deposits it receives and the net-worth accumulated in the previous periods. The amount of lending that the bank can provide is therefore determined by the household

obtains revenues from lending to goods producing firms (according to the long-term contract specified above) and repays deposits they received in the previous period from households. Deposits are assumed to pay the gross riskless rate \( \bar{R}_t \). Since the deposits and lending rates are allowed to differ, the banker may make a profit in some periods and therefore will be allowed to accumulate net-worth. The amount of lending that the bank can provide is therefore determined by the household deposits it receives and the net-worth accumulated in the previous periods.
When a banker becomes a worker (this happens with probability $1 - \alpha_b$), the accumulated net-worth it carries is then transferred lump-sum to its associated household. Only at this time is the accumulated net-wealth of the banker transformed into units of consumption and therefore raises the household’s utility. The banker, therefore, wants to maximize its expected future net-worth associated with the states of the world where it becomes a worker. Note that, since a banker can potentially remain a banker forever, the value of a bank at time $t$ depends on its net-worth in all future dates.

In the case of one-period financial contracts, Gertler and Karadi (2009) show that the banking sector is very simple to aggregate. In our case, however, we need to impose further assumptions in order to make aggregation in the banking sector feasible. In order to address the heterogeneity in net-worth and revenues across banks, we setup an insurance company that receives contributions from all banks in the economy every period. We assume that these contributions are proportional to the net wealth of each individual bank. When a banker exits the market, the insurance company uses its accumulated contributions to setup a new bank which takes over all the assets and liabilities of the exiting one and gives a lump-sum quantity equal to the net-worth of the exiting bank. By doing this, we guarantee that a representative bank exists.

The probability tree below depicts the payoffs for the household associated with different outcomes of the stochastic process that determines whether a banker becomes a worker or remains a banker:

This assumption keeps the financial intermediary from accumulating enough net-worth to finance all lending without needing to receive deposits (which are costly).

The assumption of an insurance agency implies that we have a representative bank. Hence, when presenting the bank’s problem, we can drop the index for the individual bank.

4.1 Balance sheet of the representative bank and its net-worth

At time $t+1$, the net worth which the representative bank brings forward from time $t$ to $t+1$ is given by

$$n_{t+1} = (1 - \tau) \left[ rev_t - R_t b_t \right]$$

(9)
where \( \text{rev}_t \) represents the revenues obtained in \( t \) from the credit provided in date \( t \). Revenues can be written as

\[
\text{rev}_t = \int_0^1 R_{t,i} p_i^k s_{i,t} dt
\]

\[
= (1 - \alpha_k) R_{t}^L p_i^k s_{i,t} \text{opt. in period } t
\]

\[
+ (1 - \alpha_k) \alpha_k R_{t-1}^L p_{i-1}^k \tilde{s}_{i,t-1} \text{opt. in period } t-1
\]

\[
+ (1 - \alpha_k) \alpha_k^2 R_{t-2}^L p_{i-2}^k \tilde{s}_{i,t-2} + \cdots
\]

\[
= (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_k^j R_{t-j}^L p_{t-j}^k \tilde{s}_{t-j} \text{opt. in period } t-j
\]

where \( \tilde{s}_{i,t-j} \) is the period \( t \) lending from the representative bank to firms which last reoptimized in period \( t - j \).

Additionally, given the changed timing convention:

\[
\tilde{s}_{i,t-j} = \tilde{k}_{t-j}
\]

So

\[
\text{rev}_t = (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_k^j R_{t-j}^L p_{t-j}^k \tilde{k}_{t-j}
\]

(10)

The total amount of lending provided by the representative bank in period \( t \) is

\[
\text{len}_t = (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_k^j p_{t-j}^k \tilde{k}_{t-j}
\]

(11)

The representative bank faces the following balance sheet constraint:

\[
\text{len}_t \equiv n_t + b_t
\]

(12)

That is the total amount of lending must be "financed/funded" by net-worth or deposits. Now substituting this constraint into (9) gives:

\[
n_{t+1} = (1 - \tau) [\text{rev}_t - R_t b_t]
\]

\[
= (1 - \tau) [\text{rev}_t - R_t (\text{len}_t - n_t)]
\]

\[
\uparrow
\]

\[
n_{t+1} = (1 - \tau) [\text{rev}_t - R_t \text{len}_t + R_t n_t]
\]

(13)

Hence, we only need to find recursions for \( \text{rev}_t \) and \( \text{len}_t \). This is done below.

**Recursion for \( \text{rev}_t \)**

We need to find a recursion for:

\[
\text{rev}_t = (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_k^j R_{t-j}^L p_{t-j}^k \tilde{k}_{t-j}
\]

\[
= (1 - \alpha_k) R_t^L p_t^k \tilde{k}_t + (1 - \alpha_k) \sum_{j=1}^{\infty} \alpha_k^j R_{t-j}^L p_{t-j}^k \tilde{k}_{t-j}
\]

\[9\]
\[ \text{rev}_t = (1 - \alpha_k) R^L_t p_t^k \bar{k}_t + \alpha_k \text{rev}_{t-1} \] (14)

because
\[ \text{rev}_{t-1} = (1 - \alpha_k) \sum_{j=0}^{\infty} (\alpha_k)^j R^L_{t-1-j} p_{t-1-j}^k \bar{k}_{t-1-j} \]

\[ = (1 - \alpha_k) \sum_{i=1}^{\infty} \alpha_k^{i-1} R^L_{t-i} p_{t-i}^k \bar{k}_{t-i} \]

change of index \( i = j + 1 \)
\[ \text{Recursion for } \text{len}_t \]

We need to find a recursion for:
\[ \text{len}_t = (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_k^{j} p_{t-j}^k \bar{k}_{t-j} \]

\[ = (1 - \alpha_k) p_t^k \bar{k}_t + (1 - \alpha_k) \sum_{j=1}^{\infty} \alpha_k^{j} p_{t-j}^k \bar{k}_{t-j} \]

\[ \text{len}_{t-1} = (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_k^{j} p_{t-1-j}^k \bar{k}_{t-1-j} \]

\[ = (1 - \alpha_k) \sum_{i=1}^{\infty} \alpha_k^{i-1} p_{t-i}^k \bar{k}_{t-i} \]

change of index \( i = j + 1 \)
\[ \text{len}_{t-1} = (1 - \alpha_k) \sum_{i=1}^{\infty} \alpha_k^{i} p_{t-i}^k \bar{k}_{t-i} \]

4.2 The value function for the representative bank

The intermediary’s expected discounted final equity \( V_t \) is given by:
\[ V_t = E_t \left[ (1 - \alpha_b) \beta^\frac{\lambda_{t+1}}{\lambda_t} n_{t+1} + \alpha_b (1 - \alpha_b) \beta^2 \frac{\lambda_{t+2}}{\lambda_t} n_{t+2} + \alpha_b^2 (1 - \alpha_b) \beta^3 \frac{\lambda_{t+3}}{\lambda_t} n_{t+3} + ... \right] \]

\[ = E_t \sum_{i=0}^{\infty} (1 - \alpha_b) \alpha_b^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} n_{t+i+1} \]

\[ = E_t \sum_{i=0}^{\infty} (1 - \alpha_b) \alpha_b^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} (1 - \tau) [\text{rev}_{t+i} - R_{t+i} \text{len}_{t+i} + R_{t+i} n_{t+i}] \]

Profitability of lending

From the law of motion for \( n_t \), we see that it is profitable for the bank to do lending if
\[ E_t \sum_{i=0}^{\infty} (1 - \alpha_b) \alpha_b^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} (1 - \tau) [\text{rev}_{t+i} - R_{t+i} \text{len}_{t+i}] \geq 0 \]

A stronger condition which is similar to the one in Gertler and Karadi (2009) is now derived.

The bank will lend if it is profitable in every period, i.e.
\[ E_t (1 - \alpha_b) \alpha_b^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} (1 - \tau) [\text{rev}_{t+i} - R_{t+i} \text{len}_{t+i}] \geq 0 \]

\[ \Delta \]
For one dollar of lending throughout a contract, i.e. if \( p_{t-i}^k \bar{k}_{t-j} = 1 \), then this condition reduces to

\[
E_t \left[ \lambda_{t+i+1} \left( \sum_{j=0}^{\infty} \alpha_k^j R_{t+i-j}^L - \sum_{j=0}^{\infty} \alpha_k^j R_{t+i} \right) \right] \geq 0
\]

Hence, when the bank lends out one dollar, then the discounted average lending rate should equal or exceed the deposit rate.

**Simple expression for the value function**

We now derive a recursive form for \( V_t \). Here, we follow Gertler and Karadi (2009) and consider:

\[
V_t = E_t \sum_{i=0}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} (1 - \tau) \left[ \text{rev}_{t+i} - R_{t+i} \text{len}_{t+i} \right]
+ E_t \sum_{i=0}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} (1 - \tau) \left( R_{t+i} - R_{t+i+1} \text{len}_{t+i} \right)
\]

\[
= (1 - \tau) \left\{ \text{len}_i \left( E_t \sum_{i=0}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} \left[ \frac{\text{rev}_{t+i}}{\text{len}_i} - R_{t+i} \frac{\text{len}_{t+i}}{\text{len}_i} \right] \right) + n_t \left( E_t \sum_{i=0}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} \frac{R_{t+i} \text{len}_{t+i}}{n_t} \right) \right\}
\]

\[
V_t = (1 - \tau) \left[ \text{len}_t x_{1,t} + n_t x_{2,t} \right] \quad (15)
\]

where we have defined

\[
x_{1,t} \equiv E_t \sum_{i=0}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} \left[ \frac{\text{rev}_{t+i}}{\text{len}_i} - R_{t+i} \frac{\text{len}_{t+i}}{\text{len}_i} \right]
\]

\[
x_{2,t} \equiv E_t \sum_{i=0}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} \frac{R_{t+i} \text{len}_{t+i}}{n_t}
\]

We will now derive the recursions for \( x_{1,t} \) and \( x_{2,t} \)

**Recursion for \( x_{1,t} \)**

\[
x_{1,t} = E_t \sum_{i=0}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} \left[ \frac{\text{rev}_{t+i}}{\text{len}_i} - R_{t+i} \frac{\text{len}_{t+i}}{\text{len}_i} \right]
\]

\[
= E_t (1 - \alpha_k) \beta \frac{\lambda_{t+1}}{\lambda_t} \left[ \frac{\text{rev}_{t}}{\text{len}_t} - R_{t} \frac{\text{len}_{t}}{\text{len}_t} \right] + E_t \sum_{i=1}^{\infty} (1 - \alpha_k) \alpha_k^i \beta^{i+1} \frac{\lambda_{t+i+1}}{\lambda_t} \left[ \frac{\text{rev}_{t+i}}{\text{len}_i} - R_{t+i} \frac{\text{len}_{t+i}}{\text{len}_i} \right]
\]
\[ x_{1,t} = E_t \left( 1 - \alpha_b \right) \beta^{\lambda_{t+1}} \frac{\lambda_{t+1}}{\lambda_t} \left[ \text{rev}_t - R_t \right] + E_t \left[ \alpha_b \beta x_{1,t+1} \lambda_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \right] \]

because
\[ x_{1,t+1} = E_{t+1} \sum_{i=0}^{\infty} (1 - \alpha_b) \alpha_b^{-i} \beta^{\lambda_{i+1}} \frac{\lambda_{i+1}}{\lambda_t} \left[ \text{rev}_{t+i+1} - R_{t+i+1} \lambda_{t+i+1} \frac{\lambda_{t+i+1}}{\lambda_t} \right] \]

\[ = E_{t+1} \sum_{i=0}^{\infty} (1 - \alpha_b) \alpha_b^{-i} \beta^{\lambda_{i+1}} \frac{\lambda_{i+1}}{\lambda_t} \left[ \text{rev}_{t+i} - R_{t+i} \lambda_{t+i} \frac{\lambda_{t+i}}{\lambda_t} \right] \]

Recursion for \( x_{2,t} \)
\[ x_{2,t} = E_t \sum_{i=0}^{\infty} (1 - \alpha_b) \alpha_b^{i} \beta^{\lambda_{i+1}} \frac{\lambda_{i+1}}{\lambda_t} \left[ R_{t+i} \lambda_{i+1} \frac{\lambda_{i+1}}{\lambda_t} / n_t \right] \]

\[ = (1 - \alpha_b) E_t \left[ \beta^{\lambda_{i+1}} \frac{\lambda_{i+1}}{\lambda_t} \right] R_t + E_t \sum_{i=0}^{\infty} (1 - \alpha_b) \alpha_b^{i} \beta^{\lambda_{i+1}} \frac{\lambda_{i+1}}{\lambda_t} \left[ R_{t+i} \lambda_{i+1} \frac{\lambda_{i+1}}{\lambda_t} / n_t \right] \]

\[ x_{2,t} = (1 - \alpha_b) E_t \left[ \beta^{\lambda_{i+1}} \frac{\lambda_{i+1}}{\lambda_t} \right] R_t + E_t \left[ x_{2,t+1} \alpha_b \beta^{\lambda_{i+1}} \lambda_{i+1} n_{i+1} \right] \]

4.3 The incentive constraint for the bank

To put an upper bound to the amount of lending done by the bank, we assume the presence of a costly enforcement problem. That is, at the beginning of any period \( t \) the banker can choose to divert the fraction \( \Lambda \) (alongside its net-worth) of bank deposits and instead transfer them back to the household of which he or she is a member. The cost to the banker is that the depositors can force the intermediary into bankruptcy and recover the remaining fraction \( 1 - \Lambda \) of assets.

For households to be willing to deposit savings with the bank the following incentive constraint must therefore hold:

\[ IC : \]

\[ \underbrace{V_t}_\text{banker’s loss from diverting} \geq \underbrace{\Lambda n_t}_\text{banker’s gain from diverting} \]

\[ V_t = \Lambda n_t \] (18)

Note that Gertler and Karadi (2009) are imposing the same assumption although they do not state it directly. Following Gertler and Karadi (2009) we can now substitute the expression for \( V_t \) into (18), i.e.

\[ V_t = \Lambda n_t \]

\[ (1 - \tau) [len_t x_{1,t} + n_t x_{2,t}] = \Lambda n_t \]
\[ len_{t+1} + n_t x_{2,t} = \frac{\lambda}{1-\tau} len_t \]
\[ n_t x_{2,t} = \frac{\lambda}{1-\tau} len_t - len_{t+1} \]
\[ n_t x_{2,t} = \left( \frac{\lambda}{1-\tau} - x_{1,t} \right) len_t \]
\[ \frac{len_t}{n_t} = \frac{x_{2,t}}{\frac{\lambda}{1-\tau} - x_{1,t}} \]

where we have defined \( lev_t = \frac{len_t}{n_t} = \frac{x_{2,t}}{\frac{\lambda}{1-\tau} - x_{1,t}} \) is the leverage ratio.

5 Insurance company

This section describes the revenue and costs for the insurance company. We first note that the revenue from the insurance company is

\[ rev_{t}^{ins} = \tau [rev_t - R_t len_t + R_t n_t] \]

On the other hand, the costs for the insurance company is the fraction of bankers which retire, i.e. \((1 - \alpha_b)\). Hence,

\[ cost_{t}^{ins} = (1 - \alpha_b) n_t. \]

We assume that this insurance company is owned by the households. Moreover, any periodic revenue is transferred to households.

6 Capital Producing Firms

All firms in this sector face the same problem with the same constraints, meaning that there is a representative capital-producing firm. Hence, we can leave out a subscript for firms in this section.

We assume that the capital producing firms face the following problem:

\[ \max_{k_{t+1}, k_t, i_t, v_t} profit^k = E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_t + j}{\lambda_t} \left[ v_{t+j} - v_{t+j}(1 - \omega) - i_{t+j} \right] \]

\[ \max_{k_{t+1}, k_t, i_t, v_t} profit^k = E_t \sum_{j=0}^{\infty} \beta^j \frac{\lambda_t + j}{\lambda_t} \left[ \omega v_{t+j} - i_{t+j} \right] \]

where

\[ v_t \equiv (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_k \bar{k}_{t-j} p_t^{k_{t-j}} \]

Notice in (20) that the capital producing firm buys the used capital from the firms, and the expression for \( v_t \) is a weighted average of the prices and quantities demanded in the past. The recursion for \( v_t \) is similar to the one for \( len_t \), i.e.

\[ v_t = (1 - \alpha_k) \bar{k}_t p_t^{k_{t}} + \alpha_k v_{t-1} \]
Similarly, \( q \) having more

This problem is solved subject to the constraints:

\[
k_{t+j+1} = (1 - \delta)k_{t+j} + i_{t+j} - \frac{\kappa_1}{2} \left( \frac{i_{t+j}}{i_{t+j-1}} - 1 \right)^2 i_{t+j} - \frac{\kappa_2}{2} \left( \frac{i_{t+j}}{k_{t+j}} - \frac{i_{ss}}{k_{ss}} \right)^2 k_{t+j}
\]

(23)

and the law of motion for \( \tilde{k}_t \):

\[
k_t = (1 - \alpha_k) \tilde{k}_t + \alpha_k k_{t-1}
\]

(24)

The lagrange function for this problem therefore reads

\[
L = E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} [\omega v_{t+j} - i_{t+j}]
+ E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} u_{1,t+j} \left[ (1 - \alpha_k) \tilde{k}_{t+j}^p + \alpha_k v_{t-1+j} - v_{t+j} \right]
+ E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} q_{t+j} \left[ (1 - \delta)k_{t+j} + i_{t+j} - \frac{\alpha_k}{2} \left( \frac{i_{t+j}}{i_{t+j-1}} - 1 \right)^2 i_{t+j} - \frac{\kappa_2}{2} \left( \frac{i_{t+j}}{k_{t+j}} - \frac{i_{ss}}{k_{ss}} \right)^2 k_{t+j} - k_{t+j+1} \right]
+ E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda_{t+j}}{\lambda_t} (-u_{3,t+j}) \left[ (1 - \alpha_k) \tilde{k}_{t+j} + \alpha_k k_{t-1+j} - k_{t+j} \right]
\]

We see that \( u_{1,t} \) denotes the marginal change in the lagrange function (and hence in profits) when having more \( v_t \). Similarly, \( q_t \) is the marginal change in profit of having more \( k_{t+1} \). Finally, \( -u_{3,t} \) is the marginal change in profit of having more \( k_t \).

Note that steady state profit for these firms are:

\[
profit_{ss}^k = \sum_{j=0}^{+\infty} \beta^j [\omega v_{ss} - i_{ss}]
= \frac{[\omega v_{ss} - i_{ss}]}{1 - \beta}
\]

which is positive provided \( \omega v_{ss} > i_{ss} \). Below we find that \( v_{ss} = \tilde{k}_{ss} p_{ss}^k \) and \( \tilde{k}_{ss} = k_{ss} \). So \( v_{ss} = k_{ss} p_{ss}^k \)

A non-negative profit condition is:

\[
\omega v_{ss} - i_{ss} \geq 0
\]

\[
\uparrow
\omega k_{ss} p_{ss}^k - i_{ss} \geq 0
\]

\[
\uparrow
\omega p_{ss}^k - \frac{i_{ss}}{k_{ss}} \geq 0
\]

\[
\uparrow
\omega p_{ss}^k - \delta \geq 0
\]

because \( \frac{i_{ss}}{k_{ss}} = \delta \)

\[
\uparrow
\omega \left( \frac{p_{ss}^k - \delta}{\omega} \right) \geq 0
\]

\[
\uparrow \omega \left( \frac{q_{ss}(1 - (1 - \delta))}{\beta} \frac{1 - \beta \alpha_k}{\omega} - \delta \right) \geq 0
\]

\[
\uparrow \left( \frac{q_{ss}(1 - (1 - \delta))}{\beta} (1 - \beta \alpha_k) - \delta \right) \geq 0
\]

\[
\uparrow \left( \frac{q_{ss}(1 - (1 - \delta))}{\beta} \right) - \delta \geq 0
\]

\[
\uparrow \frac{1 - \beta(1 - \delta) - \beta \delta}{\beta} \geq 0
\]

because \( q_{ss} = 1 \)

\[
\uparrow \frac{1 - \beta + \beta \delta - \beta \delta}{\beta} \geq 0
\]

\[
\uparrow \frac{1 - \beta}{\beta} \geq 0
\]
6.1 The first order conditions:

$$\frac{\partial L}{\partial v_t} = \omega - u_{1,t} + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{1,t+1} \alpha_k \right] = 0$$

\[\uparrow\]

$$u_{1,t} = \omega + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{1,t+1} \alpha_k \right]$$ (25)

Here $u_{1,t}$ is the marginal change in profit of $v_t$, i.e. $u_{1,t} = \frac{\partial \text{profit}}{\partial v_t}$ following the envelop theorem.

LHS (cost): The firm has to buy more $v_t$ in this period to sell more $v_t$.

RHS (gain): The firm earns additional $\omega$ by selling more $v_t$ in this period, and in the next period the firm is certain that it will also sell additional $\alpha_k$ units of $v_{t+1}$. The latter increases profit in the next period by $u_{1,t+1} \alpha_k$ and its net present value is thus $E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{1,t+1} \alpha_k \right]$.

$$\frac{\partial L}{\partial q_t} = -q_t + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \left( 1 - \delta \right) \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + \kappa_2 \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \left( \frac{i_{t+1}}{k_{t+1}} \right) \right] + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \alpha_k \right] = 0$$

\[\uparrow\]

$$1 = q_t \left( 1 - \frac{\kappa_1}{2} \left( \frac{i_t}{i_{t-1}} - 1 \right) \right)^2 + \kappa_1 \left( \frac{i_t}{k_t} \right) + \kappa_2 \left( \frac{i_t}{k_t} \right) + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \alpha_k \right]$$ (26)

Here $q_t$ is the marginal gain in profit from having more capital in period $t+1$, i.e. $q_t = \frac{\partial \text{profit}}{\partial k_{t+1}}$ following the envelop theorem. That is $q_t$ is Tobin’s q and $q_t = 1$ if $\kappa_1 = 0$ and $\kappa_2 = 0$. Note also that in the steady state $q_{ss} = 1$.

$$\frac{\partial L}{\partial \alpha_k} = -q_t + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \left( 1 - \delta \right) \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + \kappa_2 \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \left( \frac{i_{t+1}}{k_{t+1}} \right) \right] + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{3,t+1} \alpha_k \right] = 0$$

\[\uparrow\]

$$q_t - E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{3,t+1} \alpha_k \right] = E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \left( 1 - \delta \right) \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + \kappa_2 \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \left( \frac{i_{t+1}}{k_{t+1}} \right) \right]$$ (27)

$$- E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{3,t+1} \alpha_k \right]$$

(28)

Here, $u_{3,t}$ is the marginal gain in profit from having more capital in period $t$, i.e. $u_{3,t} = \frac{\partial \text{profit}}{\partial k_{t+1}}$ following the envelop theorem.

LHS (cost): to increase $k_{t+1}$, the firm needs to increase investment accordingly in period $t$ and this lowers profit by $q_t$ and $-E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{3,t+1} \alpha_k \right]$.

RHS (gain): an increase in $k_{t+1}$ increases capital in period $t+2$. The net present value of this increase is $E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \left( 1 - \delta - \frac{\kappa_2}{2} \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right)^2 + \kappa_2 \left( \frac{i_{t+1}}{k_{t+1}} - \frac{i_{ss}}{k_{ss}} \right) \left( \frac{i_{t+1}}{k_{t+1}} \right) \right] - E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{3,t+1} \alpha_k \right]$.

$$\frac{\partial L}{\partial \alpha_k} = u_{1,t} (1 - \alpha_k) p_t - u_{3,t} (1 - \alpha_k) = 0$$

\[\uparrow\]

$$u_{1,t} p_t^k = u_{3,t}$$ (29)

LHS: marginal gain of having more $\tilde{k}_t$ is $p_t^k$ times $u_{1,t}$, which is the marginal gain of having more $v_t$.

RHS: the marginal cost for the firm of increasing $\tilde{k}_t$ is $u_{3,t}$, which is the marginal cost of getting more capital.
6.2 Simplified formulation without adjustment costs

When we do not consider adjustment costs the problem reads:

$$\max_{k_{t+1}, k_t, v_t} \text{profit}^k_t = E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda^j}{\lambda_t} \left[ \omega v_{t+j} - i_{t+j} \right]$$

(30)

where

$$v_t \equiv (1 - \alpha_k) \sum_{j=0}^{+\infty} \alpha^j_t k_{t-j} p^k_{t-j}$$

(31)

and

$$v_t = (1 - \alpha_k) \bar{\omega}_t p^k_t + \alpha_k v_{t-1}$$

(32)

This problem is solved subject to the constraints:

$$k_{t+j+1} = (1 - \delta) k_{t+j} + i_{t+j}$$

(33)

and the law of motion for $k_t$:

$$k_t = (1 - \alpha_k) \bar{\omega}_t + \alpha_k k_{t-1}$$

(34)

The lagrange function for this problem therefore reads

$$L = E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda^j}{\lambda_t} \left[ \omega v_{t+j} - (k_{t+j+1} - (1 - \delta) k_{t+j}) \right]$$

$$+ E_t \sum_{j=0}^{+\infty} \beta^j \frac{\lambda^j}{\lambda_t} u_{1,t+j} \left[ (1 - \alpha_k) \bar{\omega}_t p^k_{t+j} + \alpha_k v_{t-1+j} - v_{t+j} \right]$$

subject to

$$k_t = (1 - \alpha_k) \bar{\omega}_t + \alpha_k k_{t-1}$$

$$k_{t+j+1} - \alpha_k k_{t-1+j} = 0$$

$$k_{t+j+1} = (1 - \delta) k_{t+j} + i_{t+j}$$

because $k_{t+j} - \alpha_k k_{t-1+j} = 0$

We see that $u_{1,t}$ denotes the marginal change in the lagrange function (and hence in profits) when having more $v_t$.

Optimization with respect to $v_t$ and $k_{t+1}$ gives

$$\frac{\partial L}{\partial v_t} = \omega - u_{1,t} + E_t \left[ \beta \frac{\lambda^j}{\lambda_t} u_{1,t+1} \alpha_k \right] = 0$$

$$\frac{\partial L}{\partial k_{t+1}} = -1 + E_t \left[ \beta \frac{\lambda^j}{\lambda_t} (1 - \delta) \right] + E_t \left[ \beta \frac{\lambda^j}{\lambda_t} u_{1,t+1} p^k_{t+1} \right] - E_t \left[ \beta \frac{\lambda^j}{\lambda_t} u_{1,t+2} p^k_{t+2} \right] = 0$$

LHS: increasing $k_{t+1}$ by one units costs 1 unit in this period. But there is also a second round cost because when the firm does not enter provide $k_{t+1}$ to good-producing firms in the next period, they also do not provide capital of $1 \times \alpha_k$ to periods into the future. Its value of this second round effect is $E_t \left[ \beta \frac{\lambda^j}{\lambda_t} u_{1,t+2} p^k_{t+2} \right]$. 

7 Market Clearing

Aggregate output in the intermediate sector is given by:
\[ y_t = \int_0^1 a_t \theta h_i^{1-\theta} \, di \]
\[ = a_t \int_0^1 \left( \frac{k_i}{n_i} \right)^{\theta} h_i \, di \]
\[ = a_t \int_0^1 \left( \frac{k_i}{n_i} \right)^{\theta} h_i \, di \]
\[ = a_t \left( \frac{k_i}{n_i} \right)^{\theta} \int_0^1 h_i \, di \]
\[ = a_t \left( \frac{k_i}{n_i} \right)^{\theta} h_i \]
\[ = a_t k_i^{\theta} h_i^{1-\theta} \]

The economy-wide resources constraint is given by:
\[ y_t = c_t + i_t \]

Finally we assume the technological shock and the preference shocks follow AR(1) processes:
\[ \log a_t = \rho_a \log a_{t-1} + \varepsilon^a_t \]
8 Summarizing

Household:
1) $\lambda_t = E_t \left( c_{t+1} - bc_{t-1} \right)^{-\sigma_c} - \beta (c_{t+1} - bc_t)^{-\sigma_c}$
2) $1 = E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} R_t \right]
3) $\phi_h = \lambda_t w_t$

Intermediate Goods Producing Firms:
4) $h_t = \left( \frac{w_t}{a_t(1-\gamma)} \right)^{-\frac{1}{\gamma}} k_t$
5) $z_{1,t} = \left( \frac{R_t}{p_t} + \omega \right) p_t \tilde{z}_{2,t}$
6) $z_{1,t} = \Theta a_t \left( \frac{w_t}{a_t(1-\gamma)} \right)^{-\frac{1}{\gamma}} + E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} z_{1,t+1} \right]
7) $z_{2,t} = 1 + E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \beta \alpha \right] z_{2,t+1}$
8) $k_t = (1 - \alpha) k_{t-1} + \alpha k_{t-1}$

Financial Intermediaries:
9) $n_{t+1} = (1 - \tau) [rev_t - R_t len_t + R_t n_t]$
10) $rev_t = (1 - \alpha_k) R_{t+1} \rho_t \kappa_t + \alpha_k \kappa_{t-1}$
11) $len_t = (1 - \alpha_k) \rho_t \kappa_t + \alpha_k \kappa_{t-1}$
12) $lev_t = \frac{len_t}{n_t} = \frac{n_t \kappa_{t-1}}{\rho_{t-1} \kappa_{t-1}}$
13) $\lambda_t = E_t \left( \frac{\lambda_{t+1}}{\lambda_t} \right) R_t + E_t \left[ \frac{\alpha \beta \lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right]$
14) $x_{1,t} = E_t \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) E_{t+1} \left[ \alpha \beta x_{1,t+1} \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right]$
15) $x_{2,t} = (1 - \alpha_b) E_t \left[ \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right] R_t + E_t \left[ x_{2,t+1} \alpha_b \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right]$

Capital Producing Firms:
16) $k_{t+1} = (1 - \delta) k_t + \beta \lambda_{t+1} \Lambda_t \kappa_t + \alpha k_{t-1}$
17) $u_{1,t} = \omega + E_t \left[ \beta \lambda_{t+1} \Lambda_t \kappa_t \alpha \right]$
18) $1 = q_t \left( 1 - \frac{\beta}{2} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \right)^2 - \frac{\kappa_1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right)$
19) $q_t = E_t \left[ \frac{\beta \lambda_{t+1} \Lambda_t \kappa_t}{\lambda_t n_t} \right] u_{3,t+1} = E_t \left[ \frac{\beta \lambda_{t+1} \Lambda_t \kappa_t}{\lambda_t n_t} \right] q_{t+1} \left( 1 - \frac{\beta}{2} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \right)^2 - \frac{\kappa_1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right) \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \frac{1}{\kappa_{t+1}} \left( \frac{\lambda_{t+1} n_{t+1}}{\lambda_t n_t} \right)$
20) $u_{1,t+1} = \theta t + \Theta \beta \lambda_{t+1} \Lambda_t \kappa_t$
21) $\lambda_t = (1 - \alpha_k) \kappa_t p_t + \alpha_k \kappa_{t-1}$

Market Clearing Conditions:
22) $y_t = \alpha_k \kappa_t h_t^{1-\theta}$
23) $y_t = c_t + \beta \lambda_{t+1} \Lambda_t \kappa_t$

Exogenous Processes:
24) $\log \alpha_t = \rho_t \log \alpha_{t-1} + \varepsilon_t$

The 24 variables in the system are as follows
$\lambda_t, c_t, R_t, h_t, w_t, k_t, z_{1,t}, z_{2,t}, p_t, \tilde{z}_t, n_t, \text{rev}_t, R_{t+1}, \text{len}_t, \text{lev}_t, x_{1,t}, x_{2,t}, i_t, u_{1,t}, q_t, u_{3,t}, a_t, v_t, y_t$

Extra output:
The average loan rate:
$R_t^{avg} = (1 - \alpha_k) \sum_{j=0}^{\infty} \alpha_j R_{t-j}$
\[ = (1 - \alpha_k) R_t^L + \sum_{j=1}^{\infty} \alpha_j^j R_{t-j}^L \]

\[ = (1 - \alpha_k) R_t^L + \alpha_k R_{t-1}^{avg, L} \]

because

\[ R_{t-1}^{avg, L} = \sum_{j=0}^{\infty} \alpha_j^j R_{t-j}^L = \sum_{i=1}^{\infty} \alpha_{i-1}^i R_{t-i}^L \quad \text{where } i = j + 1 \]

\[ \vdash \]

\[ R_{t-1}^{avg, L} \alpha_k = \sum_{i=1}^{\infty} \alpha_i^i R_{t-i}^L \]

9 The deterministic steady state

We normalize \( a_{ss} = 1 \). Note in this version, the value of \( \phi_2 \) is exogenous.

The value of \( R_{ss} \)

From EQ 2:

\[ 1 = E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} R_t \right] \]

\[ \vdash \]

\[ R_{ss} = \frac{1}{\beta} \]

The value of \( u_{1,ss} \)

From EQ 17:

\[ u_{1,t} = \omega + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} u_{1,t+1} \alpha_k \right] \]

\[ \vdash \]

\[ u_{1,ss} = \omega + \beta \alpha_k u_{1,ss} \]

\[ \vdash \]

\[ u_{1,ss} = \frac{\omega}{1 - \beta \alpha_k} \]

The value of \( q_{ss} \)

From EQ 18:

\[ 1 = q_t \left( 1 - \frac{\kappa_1}{2} \left( \frac{i_t}{i_{t-1}} - 1 \right)^2 - \kappa_1 \left( \frac{i_t}{i_{t-1}} - 1 \right) \frac{i_t}{i_{t-1}} - \kappa_2 \left( \frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right) \right) + E_t \left[ \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \kappa_1 \left( \frac{i_{t+1}}{i_t} - 1 \right) \frac{i_{ss}}{i_{ss}} \right] \]

\[ \vdash \]

\[ q_{ss} = 1 \]

The value of \( i_{ss}/k_{ss} \)

From EQ 16:

\[ k_{t+1} = (1 - \delta) k_t + i_t - \frac{\kappa_2}{2} \left( \frac{i_t}{i_{t-1}} - 1 \right)^2 i_t - \frac{\kappa_2}{2} \left( \frac{i_t}{k_t} - \frac{i_{ss}}{k_{ss}} \right)^2 k_t \]

\[ \vdash \]

\[ \frac{i_{ss}}{k_{ss}} = \delta \]

The value of \( \tilde{k}_{ss}/k_{ss} \)

From EQ 8:

\[ k_t = (1 - \alpha_k) \tilde{k}_t + \alpha_k k_{t-1} \]

\[ \vdash \]

\[ k_{ss} = (1 - \alpha_k) \tilde{k}_{ss} + \alpha_k k_{ss} \]

\[ \vdash \]
\[
\frac{k_{ss} - \alpha_k k_{ss}}{1 - \alpha_k} = \tilde{k}_{ss} \\
\Downarrow \\
\tilde{k}_{ss} = \frac{1 - \alpha_k}{1 - \alpha_k} = 1
\]

The value of \( u_{3,ss} \)

From EQ 19:

\[
q_t - E_t \left[ \beta \frac{\lambda_{s+1}}{\lambda_t} u_{3,t+1} \right] = E_t \left[ \beta \frac{\lambda_{s+1}}{\lambda_t} q_{t+1} \left( 1 - \delta \right) - \frac{\kappa_2}{2} \left( \frac{\nu_{t+1}}{k_{ss}} - \frac{l_{ss}}{k_{ss}} \right)^2 + \kappa_2 \left( \frac{\nu_{t+1}}{k_{ss}} - \frac{l_{ss}}{k_{ss}} \right) \frac{u_{s+1}}{k_{ss}} \right] - E_t \left[ \beta^2 \frac{\lambda_{s+2}}{\lambda_t} u_{3,t+2} \alpha_k \right] \\
\Downarrow \\
q_{ss} - \beta u_{3,ss} = \beta q_{ss} \left( 1 - \delta \right) - \beta^2 u_{3,ss} \alpha_k \\
\Downarrow \\
q_{ss} - \beta q_{ss} \left( 1 - \delta \right) = \beta u_{3,ss} - \beta^2 u_{3,ss} \alpha_k \\
\Downarrow \\
q_{ss} - \beta q_{ss} \left( 1 - \delta \right) = \left[ \beta - \beta^2 \alpha_k \right] u_{3,ss} \\
\Downarrow \\
u_{3,ss} = q_{ss} \frac{\left( 1 - \beta \left( 1 - \delta \right) \right)}{\beta - \beta^2 \alpha_k}
\]

The value of \( p_{ss}^k \)

From EQ 20

\[
u_{1,ss} p_{st}^k = u_{3,t} \\
\Downarrow \\
p_{ss} = \frac{u_{3,ss}}{\nu_{1,ss}}
\]

which can be expressed as

\[
p_{ss}^k = \frac{u_{3,ss}}{\nu_{1,ss}} = \frac{\frac{q_{ss} \left( 1 - \beta \left( 1 - \delta \right) \right)}{\beta - \beta^2 \alpha_k}}{\frac{\beta - \beta^2 \alpha_k}{\beta - \beta^2 \alpha_k}} = \frac{q_{ss} \left( 1 - \beta \left( 1 - \delta \right) \right)}{\beta \omega}
\]

The value of \( x_{2,ss} \)

From EQ 15:

\[
x_{2,t} = (1 - \alpha_k) E_t \left[ \beta \frac{\lambda_{s+1}}{\lambda_t} \right] R_t + E_t \left[ x_{2,t+1} \alpha_b \beta \frac{\lambda_{s+1}}{\lambda_t} \frac{n_{t+1}}{n_t} \right] \\
\Downarrow \\
x_{2,ss} = (1 - \alpha_k) \beta R_{ss} + x_{2,ss} \alpha_b \beta \\
\Downarrow \\
x_{2,ss} = \frac{(1 - \alpha_k) \beta R_{ss}}{1 - \alpha_k \beta}
\]

The value of \( \text{len}_{ss} \)

From EQ 11:

\[
\text{len}_{t} = (1 - \alpha_k) p_{st}^k \tilde{k}_{t} + \alpha_k \text{len}_{t-1} \\
\Downarrow \\
\text{len}_{ss} = (1 - \alpha_k) p_{ss}^k \tilde{k}_{ss} + \alpha_k \text{len}_{ss} \\
\Downarrow \\
\text{len}_{ss} = \frac{(1 - \alpha_k) p_{ss}^k \tilde{k}_{ss}}{1 - \alpha_k} \\
\Downarrow \\
\frac{\text{len}_{ss}}{k_{ss}} = \frac{p_{ss}^k \tilde{k}_{ss}}{k_{ss} \tilde{k}_{ss}}
\]

The value of \( r_{ss}^L \)
From EQ 10:
\[ \text{rev}_t = (1 - \alpha_k) R_{t}^{k} p_t k_t + \alpha_k \text{rev}_{t-1} \]
\[ \downarrow \]
\[ \text{rev}_{ss} = (1 - \alpha_k) R_{ss}^{k} p_{ss} k_{ss} + \alpha_k \text{rev}_{ss} \]
\[ \uparrow \]
\[ \text{rev}_{ss} = R_{ss}^{k} k_{ss} \]
\[ \downarrow \]
\[ \frac{\text{rev}_{ss}}{k_{ss}} = R_{ss}^{k} k_{ss} \]

From EQ 9:
\[ n_{t+1} = (1 - \tau) [\text{rev}_t - R_t \text{len}_t + R_t n_t] \]
\[ \downarrow \]
\[ n_{ss} = (1 - \tau) [\text{rev}_{ss} - R_{ss} \text{len}_{ss} + R_{ss} n_{ss}] \]
\[ \uparrow \]
\[ n_{ss} (1 - (1 - \tau) R_{ss}) = (1 - \tau) [\text{rev}_{ss} - R_{ss} \text{len}_{ss}] \]
\[ \downarrow \]
\[ n_{ss} = (1 - \tau) \frac{[\text{rev}_{ss} - R_{ss} \text{len}_{ss}]}{(1 - (1 - \tau) R_{ss})} \]
\[ \uparrow \]
\[ n_{ss} = \frac{(1 - \tau)}{1 - (1 - \tau) R_{ss}} \left[ \frac{\text{rev}_{ss}}{k_{ss}} - R_{ss} \frac{\text{len}_{ss}}{k_{ss}} \right] \]

From EQ 14:
\[ x_{1,t} = E_t (1 - \alpha_b) \beta \frac{\lambda_{t+1}}{\lambda_t} \left[ \frac{\text{rev}_{ss}}{\text{len}_t} - R_t \right] + E_t \left[ \alpha_b \beta x_{1,t+1} \frac{\text{len}_{t+1}}{\text{len}_t} \right] \]
\[ \downarrow \]
\[ x_{1,ss} = (1 - \alpha_b) \beta \left[ \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \right] + \alpha_b \beta x_{1,ss} \]
\[ \uparrow \]
\[ x_{1,ss} = \frac{(1 - \alpha_b) \beta}{1 - \alpha_b \beta} \left[ \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \right] \]

So finally, from EQ 12 we have
\[ \frac{\text{len}_{ss}}{n_{ss}} = \frac{x_{2,ss}}{\frac{x_{2,ss}}{x_{1,ss}} - x_{1,ss}} \]
\[ \downarrow \]
\[ \text{len}_{ss} = \left[ \frac{x_{2,ss}}{1 - \frac{x_{2,ss}}{x_{1,ss}} - x_{1,ss}} \right] n_{ss} \]
\[ \uparrow \]
\[ \text{len}_{ss} \left[ \frac{\lambda}{1 - \tau} - x_{1,ss} \right] = x_{2,ss} n_{ss} \]
\[ \downarrow \]
\[ \frac{\text{len}_{ss}}{k_{ss}} \left[ \frac{\lambda}{1 - \tau} - (1 - \alpha_b) \beta \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \right] = x_{2,ss} \frac{n_{ss}}{k_{ss}} \]
\[ \uparrow \]
\[ \frac{\text{len}_{ss}}{k_{ss}} \left[ \frac{\lambda}{1 - \tau} - (1 - \alpha_b) \beta \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \right] = x_{2,ss} \frac{(1 - \tau)}{1 - (1 - \tau) R_{ss}} \frac{\text{rev}_{ss}}{k_{ss}} - R_{ss} \frac{\text{len}_{ss}}{k_{ss}} \]

The only unknown in this equation is \( \frac{\text{rev}_{ss}}{k_{ss}} \) which we now solve for:
\[ \frac{\text{len}_{ss}}{k_{ss}} \frac{\lambda}{1 - \tau} - (1 - \alpha_b) \beta \frac{\text{rev}_{ss}}{k_{ss}} - \frac{\text{len}_{ss}}{k_{ss}} R_{ss} = x_{2,ss} \frac{(1 - \tau)}{1 - (1 - \tau) R_{ss}} \frac{\text{rev}_{ss}}{k_{ss}} - R_{ss} \frac{\text{len}_{ss}}{k_{ss}} \]
\[ \downarrow \]
\[ \frac{\text{len}_{ss}}{k_{ss}} \frac{\lambda}{1 - \tau} + (1 - \alpha_b) \beta \frac{\text{len}_{ss}}{k_{ss}} R_{ss} - (1 - \alpha_b) \beta \frac{\text{rev}_{ss}}{k_{ss}} + x_{2,ss} \frac{(1 - \tau)}{1 - (1 - \tau) R_{ss}} R_{ss} \frac{\text{len}_{ss}}{k_{ss}} = x_{2,ss} \frac{(1 - \tau)}{1 - (1 - \tau) R_{ss}} \frac{\text{rev}_{ss}}{k_{ss}} + (1 - \alpha_b) \beta \frac{\text{rev}_{ss}}{k_{ss}} \]
\[ \uparrow \]
\[ \text{rev}_{ss} \frac{\text{len}_{ss}}{k_{ss}} = \frac{x_{2,ss}}{(1 - \tau)} \frac{\lambda}{1 - \tau} + (1 - \alpha_b) \beta \frac{\text{len}_{ss}}{k_{ss}} R_{ss} + x_{2,ss} \frac{(1 - \tau)}{1 - (1 - \tau) R_{ss}} R_{ss} \frac{\text{len}_{ss}}{k_{ss}} \]
\[ \frac{(1 - \alpha_b) \beta}{1 - \alpha_b \beta} \frac{\text{rev}_{ss}}{k_{ss}} \]
Hence, we therefore get
\[ \frac{rev_{ss}}{p_{ss}} = R_{ss}^{L} \frac{E}{p_{ss}} \]
\[ R_{ss}^{L} = \frac{rev_{ss}}{p_{ss}} \frac{p_{ss}}{p_{ss}} = \frac{rev_{ss}}{p_{ss}} \]
and \( r_{ss}^{L} = R_{ss}^{L} - 1 \).

The value of \( z_{2,ss} \)

From EQ 7:
\[ z_{2,ss} = 1 + E_{t} \left[ \frac{\lambda_{t+1}}{\alpha_{k}} \beta \alpha_{k} \right] z_{2,t+1} \]
\[ z_{2,ss} = 1 + \beta \alpha_{k} z_{2,ss} \]
\[ z_{2,ss} = \frac{1}{1 - \beta \alpha_{k}} \]

The value of \( z_{1,ss} \)

From EQ5:
\[ z_{1,ss} = (r_{t}^{L} + \omega) p_{t}^{k} z_{2,t} \]
\[ z_{1,ss} = (r_{ss}^{L} + \omega) p_{ss}^{k} z_{2,ss} \]

We also note that
\[ z_{1,ss} = (r_{ss}^{L} + \omega) p_{ss}^{k} z_{2,ss} = \frac{(r_{ss}^{L} + \omega) p_{ss}^{k}}{1 - \beta \alpha_{k}} \]

The value of \( w_{ss} \)

From EQ 6:
\[ z_{1,t} = \theta a_{t} \left[ \frac{w_{t}}{a_{t}(1 - \theta)} \right]^{\frac{1-\theta}{\theta}} + E_{t} \left[ \frac{\lambda_{t+1}}{\lambda_{t}} z_{1,t+1} \right] \]
\[ z_{1,ss} = \theta a_{ss} \left[ \frac{w_{ss}}{a_{ss}(1 - \theta)} \right]^{\frac{1-\theta}{\theta}} + z_{1,ss} \alpha_{k} \beta \]
\[ z_{1,ss} = \theta a_{ss} \left[ \frac{w_{ss}}{a_{ss}(1 - \theta)} \right]^{\frac{1-\theta}{\theta}} \]
\[ \left( \frac{w_{ss}}{a_{ss}(1 - \theta)} \right)^{\frac{1-\theta}{\theta}} = \theta a_{ss} \left[ \frac{\theta a_{ss}}{(1 - \alpha_{k} \beta) z_{1,ss}} \right]^{\theta \frac{1}{\theta}} \]
\[ w_{ss} = \theta a_{ss} \left[ \frac{1}{(1 - \alpha_{k} \beta) z_{1,ss}} \right]^{\theta \frac{1}{\theta}} \]
\[ w_{ss} = (1 - \theta) \left[ \frac{\theta a_{ss}}{(1 - \alpha_{k} \beta) z_{1,ss}} \right]^{\theta \frac{1}{\theta}} \]

as \( a_{ss} = 1 \)

We then note that
\[ w_{ss} = (1 - \theta) \left[ \frac{\theta a_{ss}}{(1 - \alpha_{k} \beta) z_{1,ss}} \right]^{\theta \frac{1}{\theta}} \]
\[ = (1 - \theta) \left[ \frac{\theta}{(1 - \alpha_{k} \beta) (r_{ss}^{L} + \omega) p_{ss}^{k}} \right]^{\theta \frac{1}{\theta}} \]
\[ = (1 - \theta) \left[ \frac{\theta}{(r_{ss}^{L} + \omega) p_{ss}^{k}} \right]^{\theta \frac{1}{\theta}} \]
The value of $h_{ss}/k_{ss}$
From EQ 4:
\[ h_t = \left( \frac{w_t}{\gamma (1-\theta)} \right) \frac{1}{h_t} k_t \]
\[ \frac{h_{ss}}{k_{ss}} = \left( \frac{w_{ss}}{\gamma (1-\theta)} \right) \frac{1}{h_{ss}} k_{ss} \]

The value of $y_{ss}/k_{ss}$
From EQ 25:
\[ y_t = a_t k_t h_t^{1-\theta} \]
\[ \frac{y_{ss}}{k_{ss}} = a_{ss} k_{ss}^{\theta-1} h_{ss}^{1-\theta} \]
\[ \frac{y_{ss}}{k_{ss}} = a_{ss} \left( \frac{h_{ss}}{k_{ss}} \right)^{1-\theta} \]

The value of $c_{ss}/h_{ss}$
From EQ 22:
\[ y_t = c_t + i_t \]
\[ \frac{y_{ss}}{h_{ss}} = \frac{c_{ss}}{h_{ss}} + \frac{i_{ss}}{h_{ss}} k_{ss} \]
\[ \frac{y_{ss}}{h_{ss}} = \frac{c_{ss}}{h_{ss}} + \frac{i_{ss}}{h_{ss}} \left( \frac{h_{ss}}{k_{ss}} \right)^{-1} \]
\[ \frac{c_{ss}}{h_{ss}} = \frac{y_{ss}}{h_{ss}} - \frac{i_{ss}}{h_{ss}} \left( \frac{h_{ss}}{k_{ss}} \right)^{-1} \]

The value of $h_{ss}$
From EQ 1:
\[ \lambda_t = E_t \left[ (c_t - bc_{t-1})^{-\sigma_c} - \beta b (c_{t+1} - bc_t)^{-\sigma_c} \right] \]
\[ \lambda_{ss} = (c_{ss} - bc_{ss})^{-\sigma_c} - \beta b (c_{ss} - bc_{ss})^{-\sigma_c} \]
\[ \lambda_{ss} = c_{ss}^{-\sigma_c} (1-b)^{-\sigma_c} - \beta b c_{ss}^{-\sigma_c} (1-b)^{-\sigma_c} \]
\[ \lambda_{ss} = c_{ss}^{-\sigma_c} (1-b)^{-\sigma_c} [1 - \beta b] \]

From EQ 3:
\[ \phi_2 h_t^{\phi_1} = \lambda_t w_t \]
\[ \phi_2 h_{ss}^{\phi_1} = \lambda_{ss} w_{ss} \]
\[ \phi_2 h_{ss}^{\phi_1} = c_{ss}^{-\sigma_c} (1-b)^{-\sigma_c} [1 - \beta b] w_{ss} \]
\[ h_{ss}^{\phi_1} c_{ss}^{\sigma_c} = (1-b)^{-\sigma_c} [1 - \beta b] \frac{1}{\phi_2} w_{ss} \]
\[ h_{ss}^{\phi_1} c_{ss}^{\sigma_c} = (1-b)^{-\sigma_c} [1 - \beta b] \frac{1}{\phi_2} w_{ss} \]
\[ h_{ss}^{\phi_1} c_{ss}^{\sigma_c} = (1-b)^{-\sigma_c} [1 - \beta b] \frac{1}{\phi_2} w_{ss} \]
\[ h_{ss} = \left[ \left( \frac{w_{ss}}{k_{ss}} \right)^{-\sigma_c} (1-b)^{-\sigma_c} [1 - \beta b] \frac{1}{\phi_2} w_{ss} \right] \]
Note if we let
\[ \phi_2 = \left( \frac{c_{ss}}{h_{ss}} \right)^{-\sigma_c} (1 - b)^{-\sigma_c} [1 - \beta b] w_{ss} \]
then \( h_{ss} = 1 \). This is the normalization adopted by Marcelo.

The value of \( k_{ss} \)
\[ k_{ss} = \left( \frac{h_{ss}}{k_{ss}} \right)^{-1} \]

The value of \( c_{ss} \)
\[ c_{ss} = \left( \frac{c_{ss}}{h_{ss}} \right) h_{ss} \]

The value of \( y_{ss} \)
\[ y_{ss} = \left( \frac{y_{ss}}{h_{ss}} \right) h_{ss} \]

The value of \( \lambda_{ss} \)
\[ \lambda_{ss} = c_{ss}^{-\sigma_c} (1 - b)^{-\sigma_c} [1 - \beta b] \]

The value of \( \tilde{k}_{ss} \)
\[ \tilde{k}_{ss} = \left( \frac{\tilde{c}_{ss}}{\tilde{h}_{ss}} \right) k_{ss} \]

The value of \( i_{ss} \)
\[ i_{ss} = \left( \frac{i_{ss}}{k_{ss}} \right) k_{ss} \]

The value of \( \text{len}_{ss} \)
\[ \text{len}_{ss} = \left( \frac{\text{len}_{ss}}{k_{ss}} \right) k_{ss} \]

The value of \( n_{ss} \)
\[ n_{ss} = \left( \frac{n_{ss}}{k_{ss}} \right) k_{ss} \]

The value of \( \text{rev}_{ss} \)
\[ \text{rev}_{ss} = \left( \frac{\text{rev}_{ss}}{k_{ss}} \right) k_{ss} \]

The value of \( x_{1,ss} \)
\[ x_{1,ss} = \frac{(1 - \alpha_k)\beta}{1 - \alpha_k \beta} \left[ \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \right] \]

The value of \( V_{ss} \)
\[ V_{ss} = (1 - \tau) \left[ \text{len}_{ss} x_{1,ss} + n_{ss} x_{2,ss} \right] \]

From EQ 21:
\[ v_t = (1 - \alpha_k) \tilde{k}_{ss} p_t^k + \alpha_k v_{t-1} \]
\[ v_{ss} = (1 - \alpha_k) \tilde{k}_{ss} p_{ss}^k + \alpha_k v_{ss} \]
\[ v_{ss} = \tilde{k}_{ss} p_{ss}^k \]
10 Expression for the external finance premium and leverage

We first have that

\[ lev_t = \frac{x_{2,t}}{1 - \tau} - x_{1,t} \]

which implies

\[
lev_{ss} = \frac{x_{2,ss}}{1 - \tau} - x_{1,ss}
\]

because \( x_{2,ss} = \frac{(1 - \alpha_b)\beta R_{ss}}{1 - \alpha_b \beta} \)

\[
= \frac{\Lambda - (1 - \alpha_b)\beta R_{ss}}{1 - \tau} - x_{1,ss}
\]

because \( x_{1,ss} = \frac{(1 - \alpha_b)\beta}{1 - \alpha_b \beta} \left( \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \right) \)

Now, as \( \frac{\text{rev}_{ss}}{\text{len}_{ss}} = 1 \) we have

\[
\text{len}_{ss} = p_{ss} k_{ss}
\]

\[
\text{rev}_{ss} = R_{ss} k_{ss} k_{ss}
\]

So \( \frac{\text{rev}_{ss}}{\text{len}_{ss}} = R_{ss} k_{ss} \)

and \( R_{ss} = \frac{1}{\beta} \)

Hence,

\[
lev_{ss} = \frac{(1 - \alpha_b)}{1 - \tau} - \frac{(1 - \alpha_b)\beta R_{ss}}{1 - \alpha_b \beta} \left( \frac{R_{ss}}{1 - \tau} - R_{ss} \right)
\]

\[
= \frac{\Lambda(1 - \alpha_b) \beta}{1 - \tau} - (1 - \alpha_b)\beta R_{ss} \left( \frac{R_{ss}}{1 - \tau} - R_{ss} \right)
\]

\[
= \frac{-\Lambda(1 - \alpha_b) \beta}{(1 - \tau)(1 - \alpha_b)} \left( \frac{R_{ss}}{1 - \tau} - R_{ss} \right)
\]

From the law of motion of banks' net worth we have

\[
n_{ss} = (1 - \tau) \left( \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \text{len}_{ss} + R_{ss} n_{ss} \right)
\]

\[
\uparrow
\]

\[
n_{ss} (1 - (1 - \tau) R_{ss}) = (1 - \tau) \left( \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \text{len}_{ss} \right)
\]

\[
\uparrow
\]

\[
n_{ss} \frac{\text{len}_{ss}}{\text{len}_{ss}} (1 - (1 - \tau) R_{ss}) = (1 - \tau) \left( \frac{\text{rev}_{ss}}{\text{len}_{ss}} - R_{ss} \right)
\]

\[
\uparrow
\]

\[
\text{lev}_{ss}^{-1} = \frac{n_{ss}}{\text{len}_{ss}} = \frac{(1 - \tau) \left( \frac{R_{ss}^k}{1 - \tau} - R_{ss} \right)}{1 - (1 - \tau) R_{ss}}
\]

\[
\uparrow
\]

\[
\frac{1}{\text{lev}_{ss}} = \frac{R_{ss}^k - R_{ss}}{1 - (1 - \tau) R_{ss}} - \frac{1}{\beta}
\]

Equating the two expressions for leverage, we get

\[
\frac{\Lambda(1 - \alpha_b) \beta}{(1 - \tau)(1 - \alpha_b)} - \beta \left( \frac{R_{ss}^k}{1 - \tau} - R_{ss} \right) = \frac{R_{ss}^k - R_{ss}}{1 - (1 - \tau) R_{ss}} - \frac{1}{\beta}
\]

\[
\uparrow
\]

\[
\frac{\Lambda(1 - \alpha_b) \beta}{(1 - \tau)(1 - \alpha_b)} \left( \frac{1}{1 - \tau} - \frac{1}{\beta} \right) = R_{ss}^k - R_{ss} + \left( \frac{1}{1 - \tau} - \frac{1}{\beta} \right) \beta \left( R_{ss}^k - R_{ss} \right)
\]

\[
\uparrow
\]

\[
\frac{\Lambda(1 - \alpha_b) \beta}{(1 - \tau)(1 - \alpha_b)} \left( \frac{1}{1 - \tau} - \frac{1}{\beta} \right) = (R_{ss}^k - R_{ss}) \left[ 1 + \left( \frac{\beta}{1 - \tau} - 1 \right) \right]
\]

25
\[
R^k_{ss} - R_{ss} = \frac{\Lambda(1-\alpha_{kk})}{(1-\tau)(1-\alpha_k)} \left( \frac{1}{1-\beta} - \frac{1}{\beta} \right) \\
= \frac{\Lambda(1-\alpha_{kk})}{(1-\tau)(1-\alpha_k)} \left( \frac{1}{1-\tau} - \frac{1}{\beta} \right) \\
= \frac{\Lambda(1-\alpha_{kk})}{(1-\tau)(1-\alpha_k)} \left( \frac{1}{1-\tau} - \frac{1}{\beta} \right)
\]

For \( \beta = 1 \) we get an approximate expression:
\[
R^k_{ss} - R_{ss} = \Lambda \left( \frac{1}{1-\tau} - \frac{1}{1-\tau} \right)
\]

Interpretation:
1. The more bankers can steal (the higher \( \Lambda \)) the higher the equilibrium spread necessary to prevent them from doing so (in effect their leverage will be lower, see below, so they will have to be making lower profits)?
2. The higher the tax rate \( \tau \) and, implicitly the less profitable is banking activity, the higher the spread necessary to induce bankers to stay in business (as above this may have volume implications as well)?

The expression for leverage reads:
\[
\frac{1}{lev_{ss}} = \frac{R^k_{ss} - R_{ss}}{1-\tau - \frac{1}{\beta}}
\]
\[
\frac{1}{1-\tau} - \frac{1}{\beta} = (R^k_{ss} - R_{ss}) lev_{ss}
\]
\[
\frac{1}{1-\tau} - \frac{1}{\beta} = \frac{\Lambda(1-\alpha_{kk})}{(1-\tau)(1-\alpha_k)} \left( \frac{1}{1-\tau} - \frac{1}{\beta} \right) lev_{ss}
\]
\[
lev_{ss} = \frac{1}{\Lambda(1-\alpha_{kk})} \left( \frac{1}{1-\tau} - \frac{1}{\beta} \right)
\]
\[
lev_{ss} = \frac{1}{\Lambda(1-\alpha_{kk})}
\]
\[
lev_{ss} = \frac{\beta(1-\alpha_k)}{\Lambda(1-\alpha_{kk})}
\]

Hence for \( \beta \) close to one we have
\[
lev_{ss} = \frac{1}{\Lambda}
\]

Hence, if bankers can run away with 100 pct of lending (\( \Lambda = 1 \)), then we have no leverage. On the other hand, in the limit where they cannot run away, then we have infinite leverage. This means infinite level of credit and this drives the spread to zero. The interesting case is when \( \Lambda \) is somewhere between these intermediate cases.

Given this expression for leverage, we can write the spread as
\[
R^k_{ss} - R_{ss} = \frac{\Lambda(1-\alpha_{kk})}{\beta(1-\alpha_k)} \left( \frac{1}{1-\tau} - \frac{1}{\beta} \right)
\]
\[
= \frac{1}{lev_{ss}} \left( \frac{1}{1-\tau} - \frac{1}{\beta} \right)
\]

We also have that deposits are given by \( b_t = len_t - n_t \), meaning that
\[
b_{ss} = len_{ss} - n_{ss}
\]
\[
= len_{ss} - \frac{(1-\tau)[R^k_{ss} - R_{ss}]}{1-(1-\tau)R_{ss}} len_{ss}
\]
because \( \frac{\alpha_{ss}}{\lambda_{ss}} = \frac{(1-\tau)[R_{ss}^k - R_{ss}]}{1 - (1-\tau)R_{ss}} \)

\[ = p_{ss}^k k_{ss} \left[ 1 - \frac{(1-\tau)}{1 - (1-\tau)R_{ss}} \left[ R_{ss}^k - R_{ss} \right] \right] \]

because \( len_{ss} = p_{ss}^k k_{ss} \)

\[ = p_{ss}^k k_{ss} \left[ 1 - \frac{(1-\tau)}{1 - (1-\tau)R_{ss}} \frac{1}{\frac{1}{1-\tau}} \left( \frac{1}{1-\tau} - \frac{1}{\beta} \right) \right] \]

\[ = p_{ss}^k k_{ss} \left[ 1 - \frac{1}{1 - (1-\tau)R_{ss}} \frac{1}{\frac{1}{1-\tau}} (1 - R_{ss} (1 - \tau)) \right] \]

\[ = p_{ss}^k k_{ss} \left[ 1 - \frac{1}{\frac{1}{1-\tau}} \right] \]

11 Steady state analysis: capital level and firms profit

This section analyzes the steady state of the model in greater detail. We derive two results:

Result 1:
The capital level \( k_{ss} \) is independent of \( \alpha_k \), i.e. the degree of maturity transformation in the banking sector

Hence, the inefficiency, i.e. difference in the steady state, is unrelated to the degree of maturity transformation in the banking sector. Thus, our model implies that maturity transformation does not further enhance the efficiency lose from the presence of credit frictions.

Proof:
We first note that \( p_{ss}^k = \frac{\alpha_{ss}(1-\beta)(1-\delta)}{\beta} \) and \( len_{ss}/k_{ss} = p_{ss}^k \) is independent of \( \alpha_k \). Next, \( x_{1,ss} \) and \( x_{2,ss} \) are also independent of \( \alpha_k \), meaning that the same is the case of \( len_{ss}/n_{ss} \). Thus, \( rev_{ss}/k_{ss} \) does also not dependent on \( \alpha_k \), and therefore \( R_{ss}^L = \frac{rev_{ss}}{p_{ss}^k} \) is not affected by \( \alpha_k \). The wage level is given by \( w_{ss} = (1 - \theta) \left[ \frac{\theta}{(\theta + \omega)p_{ss}^k} \right] \frac{1}{(1-\theta)} \) which is not affected by \( \alpha_k \). Finally, \( h_{ss} = 1 \) (by normalization) and \( k_{ss} = h_{ss} \left( \frac{w_{ss}}{\alpha_{ss}(1-\theta)} \right) \) meaning that \( k_{ss} \) is independent of \( \alpha_k \). This proof the result.

Result 2:
Firms’ profit in the steady state is zero

Proof:
Aggregated firm profit reads

\[ profit_t = \sum_{j=0}^{\infty} \beta^j \frac{\alpha_{ss}}{\lambda_{ss}} \left( a_{t+j} k_{t+j}^k h_{t+j}^{1-\theta} - (r_{t+j}^L + \omega) p_{t+j}^k k_{t+j} - w_{t+j} h_{t+j} \right) \]

\[ \downarrow \]

\[ profit_{ss} = \frac{h_{ss}}{1-\beta} \left( k_{ss}^k h_{ss}^{1-\theta} - (r_{ss}^L + \omega) p_{ss}^k k_{ss} - w_{ss} h_{ss} \right) \]

\[ = \frac{h_{ss}}{1-\beta} \left( \left( \frac{w_{ss}}{1-\theta} \right)^{\frac{1}{(1-\theta)}} - (r_{ss}^L + \omega) p_{ss}^k k_{ss} \left( \frac{w_{ss}}{1-\theta} \right)^{\frac{1}{(1-\theta)}} - (1 - \theta) \left[ \frac{\theta}{(\theta + \omega)p_{ss}^k} \right] \frac{1}{(1-\theta)} \right) \]

using \( w_{ss} = (1 - \theta) \left[ \frac{\theta}{(\theta + \omega)p_{ss}^k} \right] \frac{1}{(1-\theta)} \)

using \( \frac{w_{ss}}{1-\theta} \leftrightarrow \frac{k_{ss}}{h_{ss}} = \left( \frac{w_{ss}}{1-\theta} \right)^{\frac{1}{(1-\theta)}} \)
Recall that the law of motion for banks' net worth reads

This section outlines how the proportional bank tax should be calibrated such that banks' net worth is bounded/stable.

12 Calibration of the proportional bank tax

We require that

\[
\theta \frac{\theta}{R_{ss} + \omega} \frac{\theta}{p_{ss}^k} \left( \frac{\theta}{R_{ss} + \omega} \frac{\theta}{p_{ss}^k} \right)^{1 - \theta} - (1 - \theta) \left[ \frac{\theta}{R_{ss} + \omega} \frac{\theta}{p_{ss}^k} \right]^{1 - \theta} = 0
\]

12.1 Calibration of the proportional bank tax

This section outlines how the proportional bank tax should be calibrated such that banks' net worth is bounded/stable. Recall that the law of motion for banks' net worth reads

\[
n_{t+1} = (1 - \tau) \left[ \text{rev}_t - R_t \text{len}_t + R_t n_t \right]
\]

We require that \(n_{t+1}\) is stable, meaning that is autoregressive component \((1 - \tau) R_t\) at the steady state is less than one. Hence, we must require that

\[
(1 - \tau) R_{ss} < 1
\]

\[\dagger\]

\[
(1 - \tau) < \frac{1}{R_{ss}}
\]

\[\ddagger\]

\[
(1 - \tau) < \beta
\]

\[\ddagger\text{because } R_{ss} = \frac{1}{\beta}\]

\[
1 - \beta < \tau
\]
13 Steady state comparison with the standard RBC model

This section compares the steady state of our model to the steady state in a standard RBC model. In the appendix on the standard RBC model we derive the model which reads

1. \( \lambda_t = E_t \left[ \frac{1}{(e_t - b c_{t+1})^{\gamma}} - \frac{\beta b}{(e_t + 1 - b c_t)^{\gamma}} \right] \)
2. \( \phi_t h_t^{\phi_t} = \lambda_t w_t \)
3. \( 1 = E_t \left[ \frac{\beta h_{t+1}}{h_t} (r_t^{k-1} + (1 - \delta)) \right] \)
4. \( h_t = \left( \frac{w_t}{a_t (1 - \eta)} \right)^{\frac{1}{\eta}} k_t \)
5. \( a_t \theta k_t^{\theta - 1} h_t^{1 - \theta} = r_t^{k} \)
6. \( y_t = c_t + i_t \)
7. \( k_{t+1} = (1 - \delta) k_t + i_t \)
8. \( y_t = a_t k_t^{1 - \theta} \)
9. \( \ln a_{t+1} = \rho \ln a_t + \epsilon_{t+1} \)

Key equations of this RBC model \((h_{ss} = 1)\) are

\( k_{ss}^{RBC} = \frac{1}{\beta} - (1 - \delta) \)
\( k_{ss}^{RBC} = \left[ \frac{\theta}{r_{ss}^{PSS}} \right]^{\frac{1}{1 - \eta}} \)
\( w_{ss}^{RBC} = (1 - \theta) \left[ \frac{\theta}{r_{ss}^{PSS}} \right]^{\frac{1}{1 - \eta}} \)
\( k_{ss}^{RBC} = \left( \frac{w_{ss}^{RBC}}{1 - \theta} \right)^{\frac{1}{\theta}} \)

We first note that both models imply that

\( k_{ss} = \left( \frac{w_{ss}}{1 - \theta} \right)^{\frac{1}{\theta}} \)

So we can get an idea of the size of the capital stock in both models by comparing the steady state wage level. In the model with credit friction, we have

\( w_{ss} = (1 - \theta) \left[ \frac{\theta}{(r_{ss}^{L} + \omega) p_{ss}^{k}} \right]^{\frac{1}{1 - \eta}} \)

The credit friction implies higher costs of financing a given capital stock compared to a model without credit frictions. Hence, we have \((r_{ss}^{L} + \omega) p_{ss}^{k} > k_{ss}^{RBC}\) which implies \(w_{ss}^{RBC} > w_{ss}\). The presence of a frictionless labor market implies \(k_{ss} = \left( \frac{w_{ss}}{1 - \theta} \right)^{\frac{1}{\theta}}\) in our model with credit friction and in the standard RBC model. Hence, \(k_{ss}^{RBC} > k_{ss}\). Thus, credit frictions induces an inefficient too low level of capital accumulation. This makes the marginal product of labor relatively low and explains the lower wage level with credit friction. Given \(h_{ss} = 1\) in both models, the lower capital level with credit frictions implies lower output \((y = k_{ss}^{RBC} h_{ss}^{1 - \theta})\) and lower investments \((i_{ss}/k_{ss} = \delta)\) and hence a lower consumption level.

References